

Variational principles and eigenvalue estimates for unbounded block operator matrices and applications

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Abstract

In this paper we establish variational principles, eigenvalue estimates and asymptotic formulae for eigenvalues of three different classes of unbounded block operator matrices. The results allow to characterise eigenvalues that are not necessarily located at the boundary of the spectrum. Applications to an example from magnetohydrodynamics and to Dirac operators on certain manifolds are given.

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1 Introduction

For self-adjoint operators variational principles are often used to derive eigenvalue estimates and to compare eigenvalues of different operators (see, e.g., [17], [20]). The standard variational principle however is limited to semi-bounded operators and to eigenvalues that are located at the boundary of the spectrum.

In this paper we establish variational principles for various classes of unbounded self-adjoint block operator matrices of the form

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}.$$

Such operators often arise in mathematical physics when coupled systems of (ordinary or partial) differential equations have to be studied. In these applications the entries of the corresponding block operator matrix are differential operators of different orders.

We consider three cases, the so-called “top dominant”, the “diagonal dominant”, and the “off-diagonal dominant” case depending on the position of the

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operators with smallest domain. In the case of differential operators the dominating operators are those with the highest order. Examples for top dominant block operator matrices occur in magnetohydrodynamics or astrophysics, examples for the off-diagonal case appear in quantum mechanics (e.g., Dirac operators).

The variational principles we establish use the so-called Schur complement associated with the given block operator matrix, which is formally given by $A - \lambda - B(D - \lambda)^{-1}B^*$ for λ in the resolvent set of D . The main problem when considering unbounded block operators is that this Schur complement is a priori not defined as an operator and has to be introduced via quadratic forms. Moreover, the relation between the spectrum of the block operator matrix and its Schur complement needs special consideration in some cases.

The paper is organised as follows: In Section 2 we establish the operator setting for the top dominant, the diagonal dominant, and the off-diagonal dominant case and we introduce the Schur complement by means of quadratic forms. In Section 3 we prove two kinds of variational principles for eigenvalues to the right of the spectrum of D , one in terms of the Rayleigh functional associated with the numerical range and one in terms of a functional associated with the quadratic numerical range. The latter enables us in Section 4 to derive upper and lower estimates and detailed asymptotic formulae for eigenvalues of block operator matrices. These estimates allow to compare the eigenvalues of the block operator matrix in the top dominant case with the eigenvalues of A and in the off-diagonal dominant case with the eigenvalues of BB^* . In Subsection 4.1 we apply these results to an example from magnetohydrodynamics which arises when studying a plane equilibrium layer of an ideal magnetised gravitating plasma bounded by rigid perfectly conducting planes. In Subsection 4.2 we present an application to Dirac operators on closed Riemannian spin manifolds with a warped product metric.

2 Block operator matrices and Schur complements

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We consider block operator matrices of the form

$$\mathcal{A}_0 = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \quad (2.1)$$

in the space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Throughout this paper we suppose that the following general assumptions are satisfied:

- (i) A, D are self-adjoint operators in the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively.
- (ii) B is a densely defined closed linear operator from \mathcal{H}_2 to \mathcal{H}_1 .
- (iii) A is bounded from below, D is bounded from above with $\max \sigma(D) =: d$.

The domains of the operators A, B , and D are denoted by $\mathcal{D}(A)$, $\mathcal{D}(B)$, and $\mathcal{D}(D)$, respectively. The natural domain of \mathcal{A}_0 is then given by

$$\mathcal{D}(\mathcal{A}_0) = (\mathcal{D}(A) \cap \mathcal{D}(B^*)) \oplus (\mathcal{D}(B) \cap \mathcal{D}(D)).$$

We consider three cases depending on the position of the operators with smallest domain, the so-called “top dominant”, “diagonal dominant”, and “off-diagonal dominant” case. More exactly, we assume

I. for the top dominant case:

- (T1) $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$,
- (T2) $\mathcal{D}(B) \subset \mathcal{D}(D)$ and $\mathcal{D}(B)$ is a core of D ;

II. for the diagonal dominant case:

- (D1) $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$,
- (D2) $\mathcal{D}(|D|^{1/2}) \subset \mathcal{D}(B)$;

III. for the off-diagonal dominant case:

- (O1) A and D are bounded.

Remark 2.1. The assumption $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$ implies that B^* is A -bounded with A -bound 0, and analogously for B and D .

Proof. The assertion follows from the facts that $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$ implies that B^* is $|A|^{1/2}$ -bounded and that $|A|^{1/2}$ is A -bounded with A -bound 0. \square

As a consequence of the different assumptions on the domains of the entries of \mathcal{A}_0 , the domain of \mathcal{A}_0 is given by

$$\mathcal{D}(\mathcal{A}_0) = \begin{cases} \mathcal{D}(A) \oplus \mathcal{D}(B) & \text{in case I,} \\ \mathcal{D}(A) \oplus \mathcal{D}(D) & \text{in case II,} \\ \mathcal{D}(B^*) \oplus \mathcal{D}(B) & \text{in case III.} \end{cases}$$

In any case \mathcal{A}_0 is densely defined and closable, and we denote its closure by \mathcal{A} .

In fact, in cases II and III the operator \mathcal{A}_0 is self-adjoint in \mathcal{H} and hence $\mathcal{A}_0 = \mathcal{A}$. In case III this is obvious, in case II this follows from the fact that $\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$ is $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ -bounded with relative bound 0.

In case I the operator \mathcal{A}_0 is essentially self-adjoint. Its closure is given by (cf. [2, Section 4.2], and also [1])

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} : y \in \mathcal{D}(D), x + \overline{(A - \nu)^{-1}B} y \in \mathcal{D}(A) \right\}, \\ \mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} A(x + \overline{(A - \nu)^{-1}B} y) - \nu \overline{(A - \nu)^{-1}B} y \\ B^*x + Dy \end{pmatrix}, \end{aligned}$$

where $\nu < \min \sigma(A)$, and $\overline{(A - \nu)^{-1}B}$ denotes the closure of the bounded operator $(A - \nu)^{-1}B$; here it follows from the boundedness of $B^*(A - \nu)^{-1}$ that $(A - \nu)^{-1}B$ is bounded.

Together with a block operator matrix (2.1) one usually associates the so-called Schur complements, the first of which is formally given by

$$S(\lambda) = A - \lambda - B(D - \lambda)^{-1}B^*, \quad \lambda \in \rho(D).$$

However, it may happen that the operator $S(\lambda)$ is not densely defined. To overcome this difficulty, we consider the closure \mathfrak{a} of the quadratic form (Ax, x) , which has domain $\mathcal{D}(|A|^{1/2})$ and satisfies $\mathfrak{a}[x, y] = (Ax, y)$, $\mathfrak{a}[x] = (Ax, x)$ for $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(|A|^{1/2})$. Similarly, let \mathfrak{d} be the closure of the quadratic form (Dx, x) with domain $\mathcal{D}(|D|^{1/2})$. Furthermore, set

$$\mathcal{D}_1 := \mathcal{D}(|A|^{1/2}) \cap \mathcal{D}(B^*), \quad \mathcal{D}_2 := \mathcal{D}(|D|^{1/2}). \quad (2.2)$$

Note that $\mathcal{D}_1 = \mathcal{D}(|A|^{1/2})$ in cases I and II and that $\mathcal{D}_1 = \mathcal{D}(B^*)$ in case III. In all three cases we have $\mathcal{D}(A) \subset \mathcal{D}_1 \oplus \mathcal{D}_2$ (in case I this has been shown in [2, Section 4.2]).

Now we can define a quadratic form $\mathfrak{s}(\lambda)$ for $\operatorname{Re} \lambda > d$ by

$$\mathfrak{s}(\lambda)[x, y] := \mathfrak{a}[x, y] - \lambda(x, y) - ((D - \lambda)^{-1}B^*x, B^*y), \quad x, y \in \mathcal{D}_1.$$

For the following, we recall that the spectrum of an operator function T on a domain $U \subset \mathbb{C}$ is defined by $\sigma(T) := \{\lambda \in U : 0 \in \sigma(T(\lambda))\}$. The point spectrum $\sigma_p(T)$ and the essential spectrum $\sigma_{\text{ess}}(T)$ are defined similarly, e.g., $\sigma_{\text{ess}}(T) := \{\lambda \in U : T(\lambda) \text{ is not Fredholm}\}$. For the definition of holomorphic families of operators of type (B) we refer the reader to [7, VII-§ 4.2].

Proposition 2.2. *In all three cases I, II, and III the quadratic form $\mathfrak{s}(\lambda)$ is closed and sectorial for $\operatorname{Re} \lambda > d$ with domain \mathcal{D}_1 independent of λ and hence $\mathfrak{s}(\lambda)$ defines a sectorial operator $S(\lambda)$. The domain of $S(\lambda)$ is given by*

$$\begin{aligned} \mathcal{D}(S(\lambda)) &= \left\{ x \in \mathcal{D}(|A|^{1/2}) : x - \overline{(A - \nu)^{-1}B}(D - \lambda)^{-1}B^*x \in \mathcal{D}(A) \right\} \text{ in case I,} \\ \mathcal{D}(S(\lambda)) &= \mathcal{D}(A) \text{ in case II, and} \\ \mathcal{D}(S(\lambda)) &= \{x \in \mathcal{D}(B^*) : (D - \lambda)^{-1}B^*x \in \mathcal{D}(B)\} \text{ in case III.} \end{aligned} \quad (2.3)$$

The operator function S is holomorphic of type (B) on $U := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > d\}$, and the spectra and point spectra of S and A coincide there, i.e.,

$$\sigma(A) \cap U = \sigma(S) \cap U, \quad (2.4)$$

$$\sigma_p(A) \cap U = \sigma_p(S) \cap U. \quad (2.5)$$

Proof. First we show that $\mathfrak{s}(\lambda)$ is closed and sectorial with domain \mathcal{D}_1 . The form \mathfrak{a} is closed and sectorial with domain $\mathcal{D}(\mathfrak{a}) = \mathcal{D}(|A|^{1/2})$, the form \mathfrak{t}_0 defined by $((\lambda - D)^{-1}B^*x, B^*x)$ is closable and sectorial on $\mathcal{D}(B^*)$, and for its closure \mathfrak{t} we have $\mathcal{D}(\mathfrak{t}) \supset \mathcal{D}(B^*)$. Hence the sum $\mathfrak{s}(\lambda) = \mathfrak{a} + \mathfrak{t}$ is closed and sectorial on $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}(\mathfrak{t})$ (see [7, Theorem VI.1.27] and [7, Theorem VI.1.31]). In case I and case II we have $\mathcal{D}(\mathfrak{a}) = \mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*) \subset \mathcal{D}(\mathfrak{t})$ and thus $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}(\mathfrak{t}) = \mathcal{D}(|A|^{1/2}) = \mathcal{D}_1$. In case III we have $\mathcal{D}(\mathfrak{a}) = \mathcal{H}_1$ and $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(\mathfrak{t}_0)$ since D is bounded and $\operatorname{Re} \lambda > d$, whence $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}(\mathfrak{t}) = \mathcal{D}(B^*) = \mathcal{D}_1$.

In case I all remaining assertions of the proposition were proved in [2, Proposition 4.4]. In case II, according to Remark 2.1, assumption (D1) implies that B^* is A -bounded with A -bound 0 and (D2) implies that $B(D - \lambda)^{-1}$ is bounded. Hence $B(D - \lambda)^{-1}B^*$ is A -bounded with A -bound less than 1, and so $S(\lambda)$ is m -sectorial with domain $\mathcal{D}(A)$ and self-adjoint for real λ . The assertion about the spectra follows from the Schur factorisation

$$\begin{pmatrix} A - \lambda & B \\ B^* & D - \lambda \end{pmatrix} = \begin{pmatrix} I & B(D - \lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S(\lambda) & 0 \\ 0 & D - \lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (D - \lambda)^{-1}B^* & I \end{pmatrix},$$

where the first and the last matrix on the right-hand side are bounded and boundedly invertible (which is not true for the other two cases).

It remains to consider case III. First we prove (2.3). Let x be in the set on the right-hand side of (2.3). Then for every $y \in \mathcal{D}(B^*)$ we have

$$((\lambda - D)^{-1}B^*x, B^*y) = (B(\lambda - D)^{-1}B^*x, y).$$

According to [7, Theorem VI.2.1 iii)], this implies that x is in the domain of the sectorial operator induced by the form $((\lambda - D)^{-1}B^*x, B^*x)$, and hence $x \in \mathcal{D}(S(\lambda))$. To show the converse inclusion let $x \in \mathcal{D}(S(\lambda)) \subset \mathcal{D}(\mathfrak{s}(\lambda)) = \mathcal{D}_1$. Then $x \in \mathcal{D}(B^*)$ and the form $\mathfrak{s}(\lambda)[x, y] = ((A - \lambda)x, y) + ((D - \lambda)^{-1}B^*x, B^*y)$ is bounded in y , which implies that $(D - \lambda)^{-1}B^*x \in \mathcal{D}(B)$.

The solutions of the equation $(\mathcal{A} - \lambda)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, i.e.,

$$\begin{aligned} (A - \lambda)x + By &= f, \\ B^*x + (D - \lambda)y &= 0, \end{aligned}$$

with $x \in \mathcal{D}(B^*)$, $y \in \mathcal{D}(B)$ and arbitrary $f \in \mathcal{H}_1$ are in one-to-one correspondence to the solutions of $S(\lambda)x = f$, i.e.,

$$(A - \lambda)x - B(D - \lambda)^{-1}B^*x = f$$

with $x \in \mathcal{D}(S(\lambda))$ via the relation $y = -(D - \lambda)^{-1}B^*x$. This implies (2.5) and the inclusion “ \supset ” in (2.4). In order to show the reverse inclusion in (2.4), let $\lambda \in \rho(S) \cap U$. We may assume that λ is real since \mathcal{A} is self-adjoint. It is not difficult to see that on the set $\mathcal{H}_1 \times \mathcal{D}(B(D - \lambda)^{-1})$ the inverse $\mathcal{R}(\lambda)$ of $\mathcal{A} - \lambda$ is given by

$$\begin{pmatrix} S(\lambda)^{-1} & -S(\lambda)^{-1}B(D - \lambda)^{-1} \\ -(D - \lambda)^{-1}B^*S(\lambda)^{-1} & (D - \lambda)^{-1} + (D - \lambda)^{-1}B^*S(\lambda)^{-1}B(D - \lambda)^{-1} \end{pmatrix}.$$

As $\mathcal{H}_1 \times \mathcal{D}(B(D - \lambda)^{-1})$ is dense in $\mathcal{H}_1 \times \mathcal{H}_2$, it remains to be shown that $\mathcal{R}(\lambda)$ is bounded. Since $|S(\lambda)|^{1/2}$ has domain $\mathcal{D}(\mathfrak{s}(\lambda)) = \mathcal{D}(B^*)$, the operator $(D - \lambda)^{-1}B^*|S(\lambda)|^{-1/2}$ is everywhere defined and closed, hence bounded. This implies that also $|S(\lambda)|^{-1/2}B(D - \lambda)^{-1}$ is bounded, and therefore

$$\begin{aligned} & (D - \lambda)^{-1}B^*S(\lambda)^{-1}B(D - \lambda)^{-1} \\ &= (D - \lambda)^{-1}B^*|S(\lambda)|^{-1/2} \operatorname{sign}(S(\lambda)^{-1})|S(\lambda)|^{-1/2}B(D - \lambda)^{-1} \end{aligned}$$

is bounded. In a similar way it is shown that the off-diagonal elements of the resolvent are bounded. So $\lambda \in \rho(\mathcal{A})$ and equality (2.4) is proved. \square

3 Variational principles

In this section we shall characterise eigenvalues of \mathcal{A} to the right of $d := \max \sigma(D)$ by variational principles based on the Rayleigh functional induced by the numerical range and on the functional λ_+ induced by the quadratic numerical range of \mathcal{A} (for the notion of quadratic numerical range see [12] and [11]).

However, the quadratic form corresponding to the operator \mathcal{A} which defines the Rayleigh functional need not be closable. Therefore we consider

$$\mathfrak{A} \left[\begin{pmatrix} x \\ y \end{pmatrix} \right] := \mathfrak{a}[x] + (B^*x, y) + (y, B^*x) + \mathfrak{d}[y]$$

for $x \in \mathcal{D}_1$ and $y \in \mathcal{D}_2$ (for the definition of \mathcal{D}_1 and \mathcal{D}_2 see (2.2)). Further we define the functional λ_+ by

$$\lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} := \frac{1}{2} \left(\frac{\mathfrak{a}[x]}{\|x\|^2} + \frac{\mathfrak{d}[y]}{\|y\|^2} + \sqrt{\left(\frac{\mathfrak{a}[x]}{\|x\|^2} - \frac{\mathfrak{d}[y]}{\|y\|^2} \right)^2 + 4 \frac{|(B^*x, y)|^2}{\|x\|^2 \|y\|^2}} \right) \quad (3.1)$$

for $x \in \mathcal{D}_1, y \in \mathcal{D}_2, x, y \neq 0$. Note that $\lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}$ is the larger of the two eigenvalues of the 2×2 matrix

$$\mathcal{A}_{x,y} := \begin{pmatrix} \frac{\mathfrak{a}[x]}{\|x\|^2} & \frac{(y, B^*x)}{\|x\|^2 \|y\|^2} \\ \frac{(B^*x, y)}{\|x\|^2 \|y\|^2} & \frac{\mathfrak{d}[y]}{\|y\|^2} \end{pmatrix}.$$

In the following theorem we characterise eigenvalues below the part of the essential spectrum that is to the right of d , that is, eigenvalues between d and

$$\lambda_e := \min(\sigma_{\text{ess}}(\mathcal{A}) \cap (d, \infty)). \quad (3.2)$$

To this end we define

$$\kappa_-(\lambda) := \dim \mathcal{L}_{(-\infty, 0)}(S(\lambda)), \quad \lambda \in \rho(D) \cap \mathbb{R},$$

where $\mathcal{L}_{(-\infty, 0)}(S(\lambda))$ denotes the spectral subspace of $S(\lambda)$ corresponding to the interval $(-\infty, 0)$. Hence $\kappa_-(\lambda)$, if it is finite, denotes the number of negative eigenvalues of the Schur complement $S(\lambda)$. By means of continuity arguments, it can be shown that $\kappa_-(\lambda)$ is constant on each real interval of the resolvent set $\rho(S)$ of S (see [2]). In the following \mathcal{L} always denotes a finite-dimensional subspace of \mathcal{H}_1 .

Theorem 3.1. *Assume that there exists a $\gamma \in (d, \infty)$ such that $\kappa_-(\gamma) < \infty$. Then there exists an $\alpha > d$ so that $(d, \alpha) \subset \rho(\mathcal{A})$. Set $\kappa := \kappa_-(\alpha)$, which is a finite number, and let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, be the finite or infinite sequence of the eigenvalues of \mathcal{A} in the interval (d, λ_e) , counted according to their multiplicities. Then we have*

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathcal{D}_1 \\ \dim \mathcal{L} = \kappa + n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} \max_{\substack{y \in \mathcal{D}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} = \max_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = \kappa + n - 1}} \inf_{\substack{x \in \mathcal{D}_1, x \neq 0 \\ x \perp \mathcal{L}}} \max_{\substack{y \in \mathcal{D}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.3)$$

$$= \min_{\substack{\mathcal{L} \subset \mathcal{D}_1 \\ \dim \mathcal{L} = \kappa + n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} \max_{y \in \mathcal{D}_2} \frac{\mathfrak{A} \begin{bmatrix} x \\ y \end{bmatrix}}{\| \begin{bmatrix} x \\ y \end{bmatrix} \|^2} = \max_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = \kappa + n - 1}} \inf_{\substack{x \in \mathcal{D}_1, x \neq 0 \\ x \perp \mathcal{L}}} \max_{y \in \mathcal{D}_2} \frac{\mathfrak{A} \begin{bmatrix} x \\ y \end{bmatrix}}{\| \begin{bmatrix} x \\ y \end{bmatrix} \|^2} \quad (3.4)$$

for $n = 1, 2, \dots, N$. If μ_m denotes any of the four expressions

$$\begin{aligned} & \inf_{\substack{\mathcal{L} \subset \mathcal{D}_1 \\ \dim \mathcal{L} = m}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} \sup_{\substack{y \in \mathcal{D}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}, & \sup_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = m - 1}} \inf_{\substack{x \in \mathcal{D}_1, x \neq 0 \\ x \perp \mathcal{L}}} \sup_{\substack{y \in \mathcal{D}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}, \\ & \inf_{\substack{\mathcal{L} \subset \mathcal{D}_1 \\ \dim \mathcal{L} = m}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} \sup_{y \in \mathcal{D}_2} \frac{\mathfrak{A} \begin{bmatrix} x \\ y \end{bmatrix}}{\| \begin{bmatrix} x \\ y \end{bmatrix} \|^2}, & \sup_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = m - 1}} \inf_{\substack{x \in \mathcal{D}_1, x \neq 0 \\ x \perp \mathcal{L}}} \sup_{y \in \mathcal{D}_2} \frac{\mathfrak{A} \begin{bmatrix} x \\ y \end{bmatrix}}{\| \begin{bmatrix} x \\ y \end{bmatrix} \|^2}, \end{aligned} \quad (3.5)$$

then

$$\mu_m = \begin{cases} d & \text{if } m = 1, 2, \dots, \kappa, \\ \lambda_e & \text{if } m \geq \kappa + N + 1. \end{cases} \quad (3.6)$$

Note that in (3.3) the conditions $x \neq 0$ and $y \neq 0$ can be replaced by $\|x\| = 1$ and $\|y\| = 1$, respectively, and in (3.4) the variation over y can be restricted to vectors with $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = 1$.

The first equality in (3.3) was proved in [10] under weaker assumptions; the first equality in (3.4) was proved in [3] and [5] under different assumptions.

Remark 3.2. If at least one μ_m is greater than d , then there exists a $\gamma > d$ such that $\kappa_-(\gamma)$ is finite (i.e., the assumption of Theorem 3.1 is satisfied).

Vice versa, if, as in Theorem 3.1, there exists a $\gamma > d$ so that $\kappa_-(\gamma)$ is finite, and $\dim \mathcal{H}_1 > \kappa$, then at least one μ_m is greater than d .

It should be noted that in general the index shift κ is not easy to calculate directly from its definition. Another way to determine it is the formula

$$\kappa = \max \{m \in \mathbb{N} : \mu_m = d\}. \quad (3.7)$$

The proofs of Theorem 3.1 and of Remark 3.2 will be given at the end of this section. First we need some lemmata.

Lemma 3.3. *If $\lambda \in \rho(D)$, $x \in \mathcal{D}_1$, $x \neq 0$, then for $y := -(D - \lambda)^{-1}B^*x$,*

$$\frac{\mathfrak{A} \left[\begin{pmatrix} x \\ y \end{pmatrix} \right]}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2} = \lambda + \frac{\mathfrak{s}(\lambda)[x]}{\|x\|^2}.$$

Proof. For $x \in \mathcal{D}_1$, $x \neq 0$, and $y := -(D - \lambda)^{-1}B^*x$ we have

$$\begin{aligned} \mathfrak{A} \left[\begin{pmatrix} x \\ y \end{pmatrix} \right] - \lambda \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 &= (\mathfrak{a} - \lambda)[x] - (B^*x, (D - \lambda)^{-1}B^*x) \\ &\quad - ((D - \lambda)^{-1}B^*x, B^*x) + (B^*x, (D - \lambda)^{-1}B^*x) \\ &= \mathfrak{s}(\lambda)[x]. \end{aligned} \quad \square$$

Note that if $\lambda \in \rho(D)$ is an eigenvalue of \mathcal{A} , then any corresponding eigenvector is of the form $\begin{pmatrix} x \\ -(D - \lambda)^{-1}B^*x \end{pmatrix}$ where $x \in \mathcal{D}_1$ is an eigenvector of S at λ , i.e., $S(\lambda)x = 0$.

For each $x \in \mathcal{D}_1$ we have

$$\frac{d}{d\lambda} \mathfrak{s}(\lambda)[x] = -\|x\|^2 - ((D - \lambda)^{-2}B^*x, B^*x) \leq -\|x\|^2, \quad (3.8)$$

hence $\mathfrak{s}(\cdot)[x]$ is a strictly decreasing function and $\lim_{\lambda \rightarrow +\infty} \mathfrak{s}(\lambda)[x] = -\infty$ for $x \neq 0$.

Lemma 3.4. *If $x \in \mathcal{D}_1$, $x \neq 0$, and $\mu > d$ is such that $\mathfrak{s}(\mu)[x] \leq 0$, then*

$$\frac{\mathfrak{A} \left[\begin{pmatrix} x \\ y \end{pmatrix} \right]}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2} \leq \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \leq \mu, \quad y \in \mathcal{D}_2, y \neq 0.$$

Proof. The first inequality follows from

$$\begin{aligned} \frac{\mathfrak{A} \left[\begin{pmatrix} x \\ y \end{pmatrix} \right]}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2} &= \frac{\mathfrak{a}[x] + (B^*x, y) + (y, B^*x) + \mathfrak{d}[y]}{\|x\|^2 + \|y\|^2} \\ &= \frac{\left(\mathcal{A}_{x,y} \begin{pmatrix} \|x\| \\ \|y\| \end{pmatrix}, \begin{pmatrix} \|x\| \\ \|y\| \end{pmatrix} \right)_{\mathbb{C}^2}}{\|x\|^2 + \|y\|^2} \leq \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

since $\lambda_+ \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ is the maximum of the numerical range of the 2×2 matrix $\mathcal{A}_{x,y}$.

For the proof of the second inequality assume first that $y \in \mathcal{D}(D)$, $y \neq 0$. Since $\lambda_+ := \lambda_+ \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ is an eigenvalue of the matrix $\mathcal{A}_{x,y}$, we have $\det(\mathcal{A}_{x,y} - \lambda_+) = 0$ or, equivalently,

$$(\mathfrak{a} - \lambda_+)[x]((D - \lambda_+)y, y) = |(B^*x, y)|^2. \quad (3.9)$$

If $\lambda_+ \leq d$, then the assertion is obvious. If $\lambda_+ > d$, then the right-hand side of (3.9) can be estimated by the Cauchy–Schwarz inequality with respect to the inner product $((\lambda_+ - D)^{-1} \cdot, \cdot)$:

$$\begin{aligned} |(B^*x, y)|^2 &= |((\lambda_+ - D)^{-1}B^*x, (\lambda_+ - D)y)|^2 \\ &\leq ((\lambda_+ - D)^{-1}B^*x, B^*x)((\lambda_+ - D)^{-1}(\lambda_+ - D)y, (\lambda_+ - D)y) \\ &= ((\lambda_+ - D)^{-1}B^*x, B^*x)((\lambda_+ - D)y, y). \end{aligned} \quad (3.10)$$

Hence

$$-(\mathfrak{a} - \lambda_+)[x] \leq ((\lambda_+ - D)^{-1}B^*x, B^*x),$$

which is equivalent to $\mathfrak{s}(\lambda_+)[x] \geq 0$. On the other hand, $\mathfrak{s}(\mu)[x] \leq 0$ and $\mathfrak{s}(\cdot)[x]$ is decreasing (see (3.8)), and thus $\lambda_+ \leq \mu$. Since $\lambda_+ \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ is continuous in y with respect to the graph norm of D , the inequality holds also for $y \in \mathcal{D}_2$. \square

Since $\mathfrak{s}(\cdot)[x]$ is strictly decreasing on (d, ∞) for $x \in \mathcal{D}_1$, $x \neq 0$, there exists at most one zero in this interval. If such a zero exists, we denote it by $p(x)$, otherwise we set $p(x) := -\infty$. The functional p is called generalised Rayleigh functional for the operator function S .

Lemma 3.5. *If for $x \in \mathcal{D}_1$, $x \neq 0$, the function $\mathfrak{s}(\cdot)[x]$ has a zero $p(x)$ in (d, ∞) , then*

$$p(x) = \max_{\substack{y \in \mathcal{D}_2 \\ y \neq 0}} \lambda_+ \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right) = \max_{y \in \mathcal{D}_2} \frac{\mathfrak{A} \left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right]}{\left\| \begin{smallmatrix} x \\ y \end{smallmatrix} \right\|^2}. \quad (3.11)$$

Proof. The inequalities “ \geq ” (with sup instead of max) are a consequence of Lemma 3.4 with $\mu = p(x)$ and of the fact that for $y = 0$ we have $\frac{\mathfrak{A} \left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right]}{\left\| \begin{smallmatrix} x \\ y \end{smallmatrix} \right\|^2} = \frac{\mathfrak{a}[x]}{\|x\|^2} \leq p(x)$. Together with Lemma 3.3 it follows that

$$p(x) = \frac{\mathfrak{A} \left[\begin{smallmatrix} x \\ -(D - \lambda)^{-1}B^*x \end{smallmatrix} \right]}{\left\| \begin{smallmatrix} x \\ -(D - \lambda)^{-1}B^*x \end{smallmatrix} \right\|^2} = \lambda_+ \left(\begin{smallmatrix} x \\ -(D - \lambda)^{-1}B^*x \end{smallmatrix} \right),$$

where the second equality only holds if $B^*x \neq 0$. It remains to be shown that the first maximum in (3.11) is also attained if $B^*x = 0$; indeed, in this case $\mathfrak{s}(\lambda)[x] = \mathfrak{a}[x] - \lambda\|x\|^2$ and hence

$$p(x) = \frac{\mathfrak{a}[x]}{\|x\|^2} = \lambda_+ \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$$

for every $y \in \mathcal{D}_2$, $y \neq 0$. \square

Proof of Theorem 3.1. The facts that S is a holomorphic operator function of type (B), that $\mathfrak{s}(\cdot)[x]$ is decreasing for $x \in \mathcal{D}_1$ and that there exists a $\gamma > d$ with $\kappa_-(\gamma) < \infty$ imply that all assumptions of [2, Theorem 2.1] are satisfied for the operator function S on the interval (d, ∞) (cf. also [2, Proposition 2.13]). Now Theorem 2.1 in [2] implies the existence of an interval $(d, \alpha) \subset \rho(\mathcal{A})$, that $\lambda_e > d$ (cf. [2, Lemma 2.9]), and that with the generalised Rayleigh functional p defined as above we have

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathcal{D}_1 \\ \dim \mathcal{L} = \kappa + n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} p(x) = \max_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = \kappa + n - 1}} \inf_{\substack{x \in \mathcal{D}_1, x \neq 0 \\ x \perp \mathcal{L}}} p(x)$$

for $n = 1, 2, \dots, N$ and

$$\inf_{\substack{\mathcal{L} \subset \mathcal{D}_1 \\ \dim \mathcal{L} = m}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} p(x) = \sup_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = m - 1}} \inf_{\substack{x \in \mathcal{D}_1, x \neq 0 \\ x \perp \mathcal{L}}} p(x) = \lambda_e$$

for $m \geq \kappa + N + 1$. Now the relations (3.3) and (3.4) and the second line in (3.6) follow from Lemma 3.5.

Finally, we prove that $\mu_m = d$ for $m \leq \kappa$ if μ_m denotes either the first or the third expression in (3.5); the other two cases are similar. Let $\mu \in (d, \alpha)$ be arbitrary. Since $\kappa_-(\lambda) = \kappa$ for all $\lambda \in (d, \alpha)$, there exists a subspace $\mathcal{L} \subset \mathcal{L}_{(-\infty, 0)}(S(\mu))$ with $\dim \mathcal{L} = m \leq \kappa$. For $x \in \mathcal{L}$, $x \neq 0$, we have $\mathfrak{s}(\mu)[x] \leq 0$, and hence $\frac{\mathfrak{a}(\frac{x}{y})}{\|\frac{x}{y}\|^2} \leq \lambda_+(\frac{x}{y}) \leq \mu$ for all $y \in \mathcal{D}_2$, $y \neq 0$, by Lemma 3.4. Since $\mu > d$ was arbitrary, this proves $\mu_m \leq d$. In order to show that $\mu_m \geq d$, let $x \in \mathcal{D}_1$, $x \neq 0$, be arbitrary. Then, for every $\varepsilon > 0$ we can choose a $y \in \mathcal{D}_2$, $\|y\| = 1$, so that $\mathfrak{d}[y] > d - \varepsilon$. According to (3.1) we have $\lambda_+(\frac{x}{y}) \geq \mathfrak{d}[y] > d - \varepsilon$ and for $t \in \mathbb{R}$ large enough also $\frac{\mathfrak{a}(\frac{tx}{ty})}{\|\frac{tx}{ty}\|^2} > d - \varepsilon$. Hence $\mu_m \geq d - \varepsilon$. \square

Proof of Remark 3.2. In order to prove the first claim, assume that there exists an $m \in \mathbb{N}$ such that $\mu_m > d$, where μ_m denotes the first expression in (3.5); the other cases are again similar. We have to show that there exists a $\gamma > d$ with $\kappa_-(\gamma) < \infty$. This is clear if $\dim \mathcal{H}_1 < \infty$. Otherwise, suppose that $\kappa_-(\mu) = \infty$ for all $\mu > d$. Then for every $\mu > d$ there exists a subspace $\mathcal{L} \subset \mathcal{L}_{(-\infty, 0)}(S(\mu))$ with $\dim \mathcal{L} = m$. From Lemma 3.4 it follows that $\lambda_+(\frac{x}{y}) \leq \mu$ for every $x \in \mathcal{L}$, $y \in \mathcal{D}_2$, $x, y \neq 0$. This implies that $\mu_m \leq \mu$ for all $\mu > d$, and hence $\mu_m \leq d$, a contradiction. The second claim is an immediate consequence of (3.3) and (3.4). \square

4 Eigenvalue estimates and applications

In this section we shall use the variational principle established in the previous section in order to derive upper and lower estimates for eigenvalues of \mathcal{A} and apply these estimates to an operator from magnetohydrodynamics in the first subsection and to Dirac operators on certain manifolds in the second subsection.

In any of the cases I, II, or III the following first lower estimate for eigenvalues λ_n of \mathcal{A} is an immediate consequence of Theorem 3.1.

Corollary 4.1. *If the diagonal element A of \mathcal{A} has eigenvalues $\nu_1(A) \leq \nu_2(A) \leq \dots \leq \nu_M(A)$, $M \in \mathbb{N}_0 \cup \{\infty\}$, below its essential spectrum, counted according*

to their multiplicities, then

$$\begin{aligned}\nu_n(A) &\leq d, & n = 1, 2, \dots, \kappa, \\ \nu_n(A) &\leq \lambda_{n-\kappa}, & n = \kappa + 1, \kappa + 2, \dots, \min\{\kappa + N, M\}.\end{aligned}$$

Proof. This follows directly from (3.3), the inequality $(Ax, x)/\|x\|^2 \leq \lambda_+ \left(\frac{x}{y}\right)$, and the standard variational principle for self-adjoint operators, which implies that $\nu_m(A) \leq \mu_m$ for $m = 1, 2, \dots, M$. \square

4.1 The top dominant case

In this subsection we assume that \mathcal{A} is in the top dominant case, i.e., the basic assumptions (i)–(iii) and in addition (T1), (T2) are satisfied. It follows from (T1) that there exist constants $a, b \geq 0$ such that

$$\|B^*x\|^2 \leq a\|x\|^2 + b\mathfrak{a}[x], \quad x \in \mathcal{D}(|A|^{1/2}). \quad (4.1)$$

The following theorem provides estimates for the eigenvalues λ_n of \mathcal{A} from above in terms of eigenvalues of the diagonal entry A of \mathcal{A}_0 . If D is bounded, then we can prove also estimates from below. Let $a' \in \mathbb{R}$, $b' \geq 0$ be such that

$$\|B^*x\|^2 \geq a'\|x\|^2 + b'\mathfrak{a}[x], \quad x \in \mathcal{D}(|A|^{1/2}). \quad (4.2)$$

Theorem 4.2. *Let $a, b \geq 0$ be such that (4.1) holds, and assume that there exists a $\gamma > d$ with $\kappa_-(\gamma) < \infty$. Define λ_e as in (3.2) and κ as in Theorem 3.1, and let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of \mathcal{A} in the interval (d, λ_e) . Moreover, let $\nu_1(A) \leq \nu_2(A) \leq \dots \leq \nu_M(A)$, $M \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of A below $\sigma_{\text{ess}}(A)$, and set $\nu_k(A) := \min \sigma_{\text{ess}}(A)$ for $k > M$. Then, for $n = 1, \dots, N$,*

$$\lambda_n \leq \frac{\nu_{\kappa+n}(A) + d}{2} + \sqrt{\left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^2 + b\nu_{\kappa+n}(A) + a}. \quad (4.3)$$

If D is bounded, $d' := \min \sigma(D)$ and $a' \in \mathbb{R}$, $b' \geq 0$ are such that (4.2) holds, then

$$\lambda_n \geq \frac{\nu_{\kappa+n}(A) + d'}{2} + \sqrt{\left(\frac{\nu_{\kappa+n}(A) - d'}{2}\right)^2 + (b'\nu_{\kappa+n}(A) + a')_+}, \quad (4.4)$$

where $(t)_+ := \max\{t, 0\}$ for $t \in \mathbb{R}$.

Using the inequalities (4.1) and (4.2) and the fact that (4.2) implies that

$$\|B^*x\|^2 \geq (a'\|x\|^2 + b'\mathfrak{a}[x])_+, \quad x \in \mathcal{D}(|A|^{1/2}),$$

one can prove Theorem 4.2 in a similar way as Theorem 5.1 in [10]; the details are left to the reader.

Remark 4.3. If A has compact resolvent, then $\sigma_{\text{ess}}(A) \cap (d, \infty) = \emptyset$ and for every $\gamma > d$ we have $\kappa_-(\gamma) < \infty$. This follows from [2, Theorem 4.5].

As a consequence of Theorem 4.2 we obtain the following asymptotic estimates if $\nu_k(A) \rightarrow \infty$ for $k \rightarrow \infty$:

Corollary 4.4. *Under the assumptions of Theorem 4.2, if $\nu_k(A) \rightarrow \infty$ for $k \rightarrow \infty$, we have*

$$\lambda_n \leq \nu_{\kappa+n}(A) + b + \frac{bd + a - b^2}{\nu_{\kappa+n}(A) - d} + \mathcal{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right), \quad (4.5)$$

$$\lambda_n \geq \nu_{\kappa+n}(A) + b' + \frac{b'd' + a' - b'^2}{\nu_{\kappa+n}(A) - d'} + \mathcal{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right). \quad (4.6)$$

Proof. Using (4.3) we obtain

$$\begin{aligned} \lambda_n &\leq \frac{\nu_{\kappa+n}(A) + d}{2} + \frac{\nu_{\kappa+n}(A) - d}{2} \sqrt{1 + (b\nu_{\kappa+n}(A) + a) \left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^{-2}} \\ &= \frac{\nu_{\kappa+n}(A) + d}{2} + \frac{\nu_{\kappa+n}(A) - d}{2} \left[1 + \frac{1}{2} (b\nu_{\kappa+n}(A) + a) \left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^{-2} \right. \\ &\quad \left. - \frac{1}{8} (b\nu_{\kappa+n}(A) + a)^2 \left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^{-4} + \mathcal{O}\left(\frac{1}{\nu_{\kappa+n}(A)^3}\right) \right] \\ &= \nu_{\kappa+n}(A) + \frac{b\nu_{\kappa+n}(A) + a}{\nu_{\kappa+n}(A) - d} - \frac{(b\nu_{\kappa+n}(A) + a)^2}{(\nu_{\kappa+n}(A) - d)^3} + \mathcal{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right) \\ &= \nu_{\kappa+n}(A) + b + \frac{bd + a - b^2}{\nu_{\kappa+n}(A) - d} + \mathcal{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right). \end{aligned}$$

This proves (4.5). The proof for (4.6) is similar using the fact that $(t)_+ \geq t$. \square

Example 4.5. When studying a plane equilibrium layer of an ideal magnetised gravitating plasma bounded by rigid perfectly conducting planes, one is lead to a spectral problem for a system of 3 coupled differential equations. The corresponding linear operator \mathcal{A}_0 is a $(1+2) \times (1+2)$ block operator matrix given by (cf. [1, Section 5], [15, Chapter 7.3], and [16])

$$\left(\begin{array}{c|cc} \rho_0^{-1} \mathcal{D} \rho_0 (v_a^2 + v_s^2) \mathcal{D} + k^2 v_a^2 & (\rho_0^{-1} \mathcal{D} \rho_0 (v_a^2 + v_s^2) + ig) k_{\perp} & (\rho_0^{-1} \mathcal{D} \rho_0 v_s^2 + ig) k_{\parallel} \\ \hline k_{\perp} ((v_a^2 + v_s^2) \mathcal{D} - ig) & k^2 v_a^2 + k_{\perp}^2 v_s^2 & k_{\perp} k_{\parallel} v_s^2 \\ k_{\parallel} (v_s^2 \mathcal{D} - ig) & k_{\perp} k_{\parallel} v_s^2 & k_{\parallel}^2 v_s^2 \end{array} \right)$$

in the space $L_{\rho_0}^2(0, 1) \oplus (L_{\rho_0}^2(0, 1))^2$, where $L_{\rho_0}^2(0, 1)$ denotes the L^2 -space with weight ρ_0 , \mathcal{D} the differential operator $-i \frac{d}{dx}$, $\rho_0(x)$ the equilibrium density of the plasma, $v_a(x)$ the Alfvén speed, $v_s(x)$ the sound speed, $k_{\perp}(x)$ and $k_{\parallel}(x)$ are the coordinates of the wave vector $\mathbf{k}(x)$ with respect to the field allied orthonormal bases, $k(x) = \sqrt{k_{\perp}(x)^2 + k_{\parallel}(x)^2}$ is the length of $\mathbf{k}(x)$, and g is the gravitational constant. Because the planes confining the plasma are perfectly conducting, one has to impose Dirichlet boundary conditions for the first component at $x = 0$ and $x = 1$.

The operator \mathcal{A}_0 with domain $(W_{\rho_0}^{2,2}(0, 1) \cap W_{0, \rho_0}^{1,2}(0, 1)) \oplus (W_{\rho_0}^{1,2}(0, 1))^2$ satisfies all assumptions of the top dominant case (compare [1]) and \mathcal{A}_0 is essentially self-adjoint. Here $W_{\rho_0}^{k,2}(0, 1)$ and $W_{0, \rho_0}^{k,2}(0, 1)$ denote the Sobolev space of order k associated with $L_{\rho_0}^2(0, 1)$ without and with Dirichlet boundary conditions, respectively.

In addition, in [1] it was shown that the essential spectrum of the closure \mathcal{A} of \mathcal{A}_0 is given by

$$\sigma_{\text{ess}}(\mathcal{A}) = \lambda_{\text{ak}}([0, 1]) \cup \lambda_{\text{tk}}([0, 1]),$$

i.e., the union of the ranges of the functions λ_{ak} and λ_{tk} given by the squares of the Alfvén and mean frequencies:

$$\lambda_{ak} := v_a^2 k_{\parallel}^2, \quad \lambda_{tk} := \frac{v_a^2 v_s^2}{v_a^2 + v_s^2} k_{\parallel}^2.$$

Integration by parts shows that

$$\begin{aligned} \mathfrak{a}[y] &= \int_0^1 \rho_0 p_1 |y'|^2 dx + \int_0^1 \rho_0 q_1 |y|^2 dx, \\ \|B^* y\|^2 &= \int_0^1 \rho_0 p_2 |y'|^2 dx + \int_0^1 \rho_0 q_2 |y|^2 dx, \end{aligned}$$

where the functions p_1 , q_1 , p_2 , and q_2 are given by

$$\begin{aligned} p_1 &:= v_a^2 + v_s^2, & q_1 &:= k^2 v_a^2, \\ p_2 &:= (v_a^2 + v_s^2)^2 k_{\perp}^2 + v_s^4 k_{\parallel}^2, & q_2 &:= k^2 g^2 - \frac{g}{\rho_0} \left(\rho_0 ((v_a^2 + v_s^2) k_{\perp} + v_s^2 k_{\parallel}) \right)'. \end{aligned}$$

If we set

$$b := \max \frac{p_2}{p_1}, \quad a := \max \{ \max q_2 - b \min q_1, 0 \}, \quad (4.7)$$

$$b' := \min \frac{p_2}{p_1}, \quad a' := \min q_2 - b' \max q_1, \quad (4.8)$$

then (4.1) and (4.2) are satisfied. For instance, (4.1) follows from

$$\begin{aligned} \|B^* y\|^2 &= \int_0^1 \rho_0 p_2 |y'|^2 dx + \int_0^1 \rho_0 q_2 |y|^2 dx \\ &\leq b \int_0^1 \rho_0 p_1 |y'|^2 dx + \int_0^1 \rho_0 q_2 |y|^2 dx \\ &= b \left(\int_0^1 \rho_0 p_1 |y'|^2 dx + \int_0^1 \rho_0 q_1 |y|^2 dx \right) + \int_0^1 \rho_0 (-b q_1 + q_2) |y|^2 dx \\ &\leq b \mathfrak{a}[y] + a \|y\|^2. \end{aligned}$$

To calculate the constants $d = \max \sigma(D)$ and $d' = \min \sigma(D)$, we need the spectrum of the right lower corner D of the given block operator matrix \mathcal{A} . The spectrum of this 2×2 matrix multiplication operator is the range of the functions

$$f_{\pm} := \frac{k^2(v_a^2 + v_s^2)}{2} \pm \sqrt{\frac{k^4(v_a^2 + v_s^2)^2}{4} - k^2 k_{\parallel}^2 v_a^2 v_s^2}$$

(note that $f_{\pm}(x)$ are the eigenvalues of the matrix D for a fixed x), and hence

$$d = \max f_+, \quad d' = \min f_-. \quad (4.9)$$

Theorem 4.6. *The spectrum of the operator \mathcal{A} in Example 4.5 in the interval $(\max f_+, \infty)$ consists of eigenvalues $\lambda_1 < \lambda_2 < \dots$ which accumulate at infinity. For the eigenvalues λ_n , $n = 1, 2, \dots$, the estimates (4.3), (4.4) and the*

asymptotic estimates (4.5), (4.6) hold where $\nu_n := \nu_n(A)$, $n = 1, 2, \dots$, are the eigenvalues of the operator

$$\rho_0^{-1} D \rho_0 (v_a^2 + v_s^2) D + k^2 v_a^2$$

in the space $L_{\rho_0}^2(0, 1)$ with Dirichlet boundary conditions in increasing order, the constants a, a', b, b', d , and d' are given by (4.7), (4.8), and (4.9), and κ can be calculated from (3.7). In particular, we have

$$\begin{aligned} \lambda_n &\leq \nu_{\kappa+n} + \max \frac{(v_a^2 + v_s^2)^2 k_{\perp}^2 + v_s^4 k_{\parallel}^2}{v_a^2 + v_s^2} + O\left(\frac{1}{\nu_{\kappa+n}}\right), \\ \lambda_n &\geq \nu_{\kappa+n} + \min \frac{(v_a^2 + v_s^2)^2 k_{\perp}^2 + v_s^4 k_{\parallel}^2}{v_a^2 + v_s^2} + O\left(\frac{1}{\nu_{\kappa+n}}\right). \end{aligned}$$

4.2 The off-diagonal dominant case

In this subsection we assume that \mathcal{A} is in the off-diagonal dominant case, i.e., the basic assumptions (i)–(iii) and in addition (O1) are satisfied. The following theorem provides estimates from above and below for the eigenvalues λ_n of \mathcal{A} in terms of eigenvalues of the operator BB^* .

Theorem 4.7. *Assume that there exists a $\gamma > d$ such that $\kappa_-(\gamma) < \infty$, and define λ_e as in (3.2) and κ as in Theorem 3.1. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of \mathcal{A} in the interval (d, λ_e) . Moreover, let $\nu_1((BB^*)^{1/2}) \leq \nu_2((BB^*)^{1/2}) \leq \dots \leq \nu_M((BB^*)^{1/2})$, $M \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of $(BB^*)^{1/2}$ below $\sigma_{\text{ess}}((BB^*)^{1/2})$ and set $\nu_k((BB^*)^{1/2}) := \min \sigma_{\text{ess}}((BB^*)^{1/2})$ for $k > M$. Then, for $n = 1, \dots, N$,*

$$\begin{aligned} \lambda_n &\geq \frac{\min \sigma(A) + \min \sigma(D)}{2} + \sqrt{\left(\frac{\min \sigma(A) - \min \sigma(D)}{2}\right)^2 + \nu_{\kappa+n}((BB^*)^{1/2})^2}, \\ \lambda_n &\leq \frac{\max \sigma(A) + \max \sigma(D)}{2} + \sqrt{\left(\frac{\max \sigma(A) - \max \sigma(D)}{2}\right)^2 + \nu_{\kappa+n}((BB^*)^{1/2})^2}. \end{aligned}$$

Proof. Note that the function

$$f(s, t) := s + t + \sqrt{(s - t)^2 + b}$$

is increasing in s and t for non-negative b . Thus for $x \in \mathcal{D}(B^*)$, $\|x\| = 1$, with $B^*x \neq 0$ we have

$$\begin{aligned} &\lambda_+ \left(\begin{array}{c} x \\ B^*x \end{array} \right) \\ &= \frac{1}{2} \left[(Ax, x) + \frac{(DB^*x, B^*x)}{\|B^*x\|^2} + \sqrt{\left((Ax, x) - \frac{(DB^*x, B^*x)}{\|B^*x\|^2} \right)^2 + 4\|B^*x\|^2} \right] \\ &\geq \frac{\min \sigma(A) + \min \sigma(D)}{2} + \sqrt{\left(\frac{\min \sigma(A) - \min \sigma(D)}{2} \right)^2 + \|B^*x\|^2}. \end{aligned}$$

Now $\|B^*x\|^2$ is the closure of the quadratic form (BB^*x, x) . Then the standard variational principle for BB^* implies that for every $\mathcal{L} \subset \mathcal{D}(B^*)$ with $\dim \mathcal{L} = \kappa + n$ there exists an $x_{\mathcal{L}} \in \mathcal{L}$, $\|x_{\mathcal{L}}\| = 1$, with $\|B^*x_{\mathcal{L}}\|^2 \geq \nu_{\kappa+n}((BB^*)^{1/2})^2$. Hence

$$\begin{aligned} \lambda_n &= \min_{\substack{\mathcal{L} \subset \mathcal{D}(B^*) \\ \dim \mathcal{L} = \kappa+n}} \max_{\substack{x \in \mathcal{L} \\ \|x\|=1}} \max_{\substack{y \in \mathcal{J}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \\ &\geq \min_{\substack{\mathcal{L} \subset \mathcal{D}(B^*) \\ \dim \mathcal{L} = \kappa+n}} \lambda_+ \begin{pmatrix} x_{\mathcal{L}} \\ B^*x_{\mathcal{L}} \end{pmatrix} \\ &\geq \frac{\min \sigma(A) + \min \sigma(D)}{2} + \sqrt{\left(\frac{\min \sigma(A) - \min \sigma(D)}{2}\right)^2 + \nu_{\kappa+n}((BB^*)^{1/2})^2}. \end{aligned}$$

To prove the second inequality let $\varepsilon > 0$ be arbitrary. There exists an $\mathcal{L}_\varepsilon \subset \mathcal{L}_{(-\infty, \nu_{\kappa+n}((BB^*)^{1/2})^2 + \varepsilon]}(BB^*)$ with $\dim \mathcal{L}_\varepsilon = \kappa + n$. If $x \in \mathcal{L}_\varepsilon$, then $\|B^*x\|^2 \leq \nu_{\kappa+n}((BB^*)^{1/2})^2 + \varepsilon$. Using again the monotonicity of f we conclude

$$\begin{aligned} \lambda_n &= \min_{\substack{\mathcal{L} \subset \mathcal{D}(B^*) \\ \dim \mathcal{L} = \kappa+n}} \max_{\substack{x \in \mathcal{L} \\ \|x\|=1}} \max_{\substack{y \in \mathcal{J}_2 \\ \|y\|=1}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \\ &\leq \max_{\substack{x \in \mathcal{L}_\varepsilon \\ \|x\|=1}} \max_{\substack{y \in \mathcal{J}_2 \\ \|y\|=1}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \\ &\leq \max_{\substack{x \in \mathcal{L}_\varepsilon \\ \|x\|=1}} \frac{(Ax, x) + \max \sigma(D)}{2} + \sqrt{\left(\frac{(Ax, x) - \max \sigma(D)}{2}\right)^2 + \|B^*x\|^2} \\ &\leq \frac{\max \sigma(A) + \max \sigma(D)}{2} + \sqrt{\left(\frac{\max \sigma(A) - \max \sigma(D)}{2}\right)^2 + \nu_{\kappa+n}((BB^*)^{1/2})^2 + \varepsilon}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the desired inequality follows. \square

The following particular case will be needed for the example at the end of this section.

Corollary 4.8. *Suppose that A_0 is of the form*

$$A_0 = \begin{pmatrix} A_1 & B \\ B^* & -A_2 \end{pmatrix},$$

where A_1 and A_2 are a strictly positive bounded operators with

$$\begin{aligned} a_- &:= \min \sigma(A_1) = \min \sigma(A_2), \\ a_+ &:= \max \sigma(A_1) = \max \sigma(A_2). \end{aligned}$$

Assume that both BB^* and B^*B have compact resolvents and let $\nu_{1,n}((BB^*)^{1/2})$ and $\nu_{2,n}((B^*B)^{1/2})$, $n = 1, 2, \dots$, be the eigenvalues of $(BB^*)^{1/2}$ and $(B^*B)^{1/2}$, respectively, enumerated non-increasingly. Then

$$(-a_-, a_-) \cap \sigma(\mathcal{A}) = \emptyset \tag{4.10}$$

and the spectrum of \mathcal{A} consists only of eigenvalues of finite algebraic multiplicity accumulating at most at infinity. If we denote these eigenvalues by $\dots \leq \lambda_{-2} \leq$

$\lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \dots$, then, for $n = 1, 2, \dots$,

$$\lambda_n \geq -\frac{a_+ - a_-}{2} + \sqrt{\left(\frac{a_+ + a_-}{2}\right)^2 + \nu_{1,n}((BB^*)^{1/2})^2}, \quad (4.11)$$

$$\lambda_n \leq \frac{a_+ - a_-}{2} + \sqrt{\left(\frac{a_+ + a_-}{2}\right)^2 + \nu_{1,n}((BB^*)^{1/2})^2}, \quad (4.12)$$

and, for $n = -1, -2, \dots$,

$$\lambda_n \geq -\frac{a_+ - a_-}{2} - \sqrt{\left(\frac{a_+ + a_-}{2}\right)^2 + \nu_{2,-n}((B^*B)^{1/2})^2}, \quad (4.13)$$

$$\lambda_n \leq \frac{a_+ - a_-}{2} - \sqrt{\left(\frac{a_+ + a_-}{2}\right)^2 + \nu_{2,-n}((B^*B)^{1/2})^2}. \quad (4.14)$$

Note that the assumption that both BB^* and B^*B have compact resolvents cannot be simplified in general; only in the case when B has a bounded inverse, it is equivalent to the fact that B has compact resolvent.

Proof. Since the Schur complement $S(0) = A_1 + BA_2^{-1}B^*$ is strictly positive, we can choose $\gamma = 0$ and $\alpha = 0$ in Theorem 3.1, which implies $\kappa = 0$. Now (4.10), (4.11), and (4.12) follow easily from Theorem 4.7 which in particular implies that $\lambda_+(x/y) \geq a_-$. The estimates for the negative eigenvalues follow by considering $-\mathcal{A}$ and swapping \mathcal{H}_1 and \mathcal{H}_2 . \square

From (4.11) and (4.12) it is easy to derive the following asymptotic estimates:

Corollary 4.9. *Under the assumptions of Corollary 4.8 we have, for $n \rightarrow \infty$,*

$$\begin{aligned} \lambda_n &\geq \nu_{1,n}((BB^*)^{1/2}) - \frac{a_+ - a_-}{2} + \frac{1}{2\nu_{1,n}((BB^*)^{1/2})} \left(\frac{a_+ + a_-}{2}\right)^2 \\ &\quad + O\left(\frac{1}{\nu_{1,n}((BB^*)^{1/2})^2}\right), \\ \lambda_n &\leq \nu_{1,n}((BB^*)^{1/2}) + \frac{a_+ - a_-}{2} + \frac{1}{2\nu_{1,n}((BB^*)^{1/2})} \left(\frac{a_+ + a_-}{2}\right)^2 \\ &\quad + O\left(\frac{1}{\nu_{1,n}((BB^*)^{1/2})^2}\right). \end{aligned}$$

Remark 4.10. If, in the situation of Corollary 4.8, $x_0 \in \ker B^*$, $x_0 \neq 0$, we can improve the upper bound for the first positive eigenvalue λ_1 of \mathcal{A} ; in this case we obtain from (3.3) that

$$\lambda_1 = \min_{\substack{x \in \mathcal{D}(B^*) \\ x \neq 0}} \max_{\substack{y \in \mathcal{H}_2 \\ y \neq 0}} \lambda_+\left(\frac{x}{y}\right) \leq \max_{\substack{y \in \mathcal{H}_2 \\ y \neq 0}} \lambda_+\left(\frac{x_0}{y}\right) = \frac{(A_1 x_0, x_0)}{\|x_0\|^2},$$

and hence

$$a_- \leq \lambda_1 \leq \frac{(A_1 x_0, x_0)}{\|x_0\|^2}.$$

A typical situation when Corollary 4.8 applies is when A_1 and A_2 are bounded multiplication operators and B is a regular differential operator.

Example 4.11. Let g be a positive function on $[0, 1]$. Then for the operator

$$\begin{pmatrix} g & -d/dx \\ d/dx & -g \end{pmatrix}$$

in $L^2(0, 1) \oplus L^2(0, 1)$ with periodic boundary conditions in both components we have $\nu_{1,n}((BB^*)^{1/2}) = 2\pi n$, $n = 1, 2, \dots$, and hence

$$\begin{aligned} \lambda_n &\geq 2\pi n - \frac{\max g - \min g}{2} + \frac{1}{4\pi n} \left(\frac{\max g + \min g}{2} \right)^2 + O\left(\frac{1}{n^2}\right), \\ \lambda_n &\leq 2\pi n + \frac{\max g - \min g}{2} + \frac{1}{4\pi n} \left(\frac{\max g + \min g}{2} \right)^2 + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Example 4.12. We consider the Dirac operator $D_{\mathcal{M}}$ on a closed Riemannian spin manifold \mathcal{M} with a warped product metric. These manifolds are complete Riemannian spin manifolds and hence $D_{\mathcal{M}}$ is an essentially self-adjoint operator acting on the space of spinors $\Gamma\Sigma_{\mathcal{M}}$, i.e., on sections of a certain $2^{\lfloor \frac{\dim \mathcal{M}}{2} \rfloor}$ -dimensional complex vector bundle, the so-called spinor bundle $\Sigma_{\mathcal{M}} \rightarrow \mathcal{M}$. Since the manifold \mathcal{M} is closed, the Dirac operator $D_{\mathcal{M}}$ has discrete spectrum. The kernel is not a topological but a conformal invariant and only few estimates are known for the first positive eigenvalue. In the case of an even dimensional spin manifold the spectrum of the corresponding Dirac operator is symmetric. For details on Dirac operators on manifolds we refer the reader to [14] and [4].

The manifold \mathcal{M} with its warped product metric is defined as follows. Let $(\mathcal{B}^m, g_{\mathcal{B}})$, $(\mathcal{F}^k, g_{\mathcal{F}})$ be closed Riemannian spin manifolds of dimensions m and k , respectively. Sometimes we write \mathcal{B} and \mathcal{F} to shorten the notation. For any positive C^∞ -function $f : \mathcal{B}^m \rightarrow \mathbb{R}^+$ we denote by $\mathcal{M} := \mathcal{B}^m \times_f \mathcal{F}^k := (\mathcal{B}^m \times \mathcal{F}^k, g_{\mathcal{B}} + f^2 g_{\mathcal{F}})$ the warped product of \mathcal{B}^m and \mathcal{F}^k with the product spin structure. For the spinor bundles we have

$$\begin{aligned} \Sigma_{\mathcal{M}} &\cong \pi_{\mathcal{B}}^* \Sigma_{\mathcal{B}} \otimes \pi_{\mathcal{F}}^* \Sigma_{\mathcal{F}} \quad \text{for } m \text{ or } k \text{ even,} \\ \Sigma_{\mathcal{M}} &\cong (\pi_{\mathcal{B}}^* \Sigma_{\mathcal{B}} \otimes \pi_{\mathcal{F}}^* \Sigma_{\mathcal{F}}) \oplus (\pi_{\mathcal{B}}^* \widehat{\Sigma}_{\mathcal{B}} \otimes \pi_{\mathcal{F}}^* \Sigma_{\mathcal{F}}) \quad \text{for } m \text{ and } k \text{ odd,} \end{aligned}$$

where $\pi_{\mathcal{X}} : \mathcal{M} \rightarrow \mathcal{X}$ denotes the projection, $\Sigma_{\mathcal{X}}$ the spinor bundle over \mathcal{X} for any manifold \mathcal{X} and $\Sigma_{\mathcal{B}}$ and $\widehat{\Sigma}_{\mathcal{B}}$ the two spinor bundles given by the two representations of the n -dimensional Clifford algebra Cl_n . In the case that the spin manifold \mathcal{B} is of even dimension, there is a natural splitting $\Sigma_{\mathcal{B}} = \Sigma_{\mathcal{B}}^+ \oplus \Sigma_{\mathcal{B}}^-$ and with respect to this decomposition the Dirac operator $D_{\mathcal{B}}$ has the form

$$D_{\mathcal{B}} = \begin{pmatrix} 0 & D_{\mathcal{B}}^+ \\ D_{\mathcal{B}}^- & 0 \end{pmatrix},$$

i.e., it exchanges the positive and negative spinors. An analogous representation holds for the Dirac operator $D_{\mathcal{F}}$ on \mathcal{F} if \mathcal{F} is even-dimensional.

The warped product structure of the manifold \mathcal{M} allows us to write the Dirac operator $D_{\mathcal{M}}$ as a direct sum of off-diagonal dominant block operator matrices. To this end, we decompose the space of spinors over \mathcal{M} along the eigenspaces

of the Dirac operator on the fibre \mathcal{F} . More exactly, for $\Lambda \in \sigma_p(D_{\mathcal{F}})$ let $\mathcal{E}_{\Lambda} \rightarrow \mathcal{B}$ be the vector bundle with fibre

$$\mathcal{E}_{\Lambda,b} := E\left(\frac{\Lambda}{f(b)}, D_{f(b)\mathcal{F}}\right),$$

trivialised by $(\frac{\pi_{\mathcal{F}}^* e_{\Lambda,1}}{f^{k/2}}, \dots, \frac{\pi_{\mathcal{F}}^* e_{\Lambda,r(\Lambda)}}{f^{k/2}})$ where $(e_{\Lambda,1}, \dots, e_{\Lambda,r(\Lambda)})$ is an orthonormal basis of the eigenspace $E(\Lambda, D_{\mathcal{F}})$, and $r(\Lambda)$ is the multiplicity of Λ .

Definition 4.13. For $\Lambda \in \sigma(D_{\mathcal{F}})$ define

$$W_{\Lambda} := \begin{cases} \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes \mathcal{E}_{\Lambda}) = \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}) & \text{if } m \text{ even,} \\ \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes (\mathcal{E}_{\Lambda} \oplus \mathcal{E}_{-\Lambda})) = \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{2r(\Lambda)}) & \text{if } m \text{ odd, } k \text{ even, } \Lambda \neq 0, \\ \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes \mathcal{E}_0) = \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r(0)}) & \text{if } m \text{ odd, } k \text{ even, } \Lambda = 0, \\ \Gamma_{\mathcal{B}}\left((\Sigma_{\mathcal{B}}^d \otimes \mathcal{E}_{\Lambda}) \oplus (\Sigma_{\mathcal{B}}^{od} \otimes \mathcal{E}_{\Lambda})\right) \\ = \Gamma_{\mathcal{B}}\left((\Sigma_{\mathcal{B}}^d \otimes \mathbb{C}^{r(\Lambda)}) \oplus (\Sigma_{\mathcal{B}}^{od} \otimes \mathbb{C}^{r(\Lambda)})\right) & \text{if } m, k \text{ odd,} \end{cases}$$

where $\Sigma_{\mathcal{B}}^d$ and $\Sigma_{\mathcal{B}}^{od}$ are subbundles of $\Sigma_{\mathcal{B}} \oplus \widehat{\Sigma}_{\mathcal{B}}$ given by

$$\Sigma_{\mathcal{B}}^d := \left\{ \begin{pmatrix} \varphi \\ \eta(\varphi) \end{pmatrix} : \varphi \in \Sigma_{\mathcal{B}} \right\}, \quad \Sigma_{\mathcal{B}}^{od} := \left\{ \begin{pmatrix} \varphi \\ -\eta(\varphi) \end{pmatrix} : \varphi \in \Sigma_{\mathcal{B}} \right\},$$

and $\eta: \Sigma_{\mathcal{B}} \rightarrow \widehat{\Sigma}_{\mathcal{B}}$ is the canonical isomorphism. A spinor $\Psi \in \Gamma_{\mathcal{M}}(\Sigma_{\mathcal{M}})$ is called a *spinor of weight Λ* if $\Psi \in W_{\Lambda}$.

The space of spinors decomposes as

$$\Gamma_{\mathcal{M}}(\Sigma_{\mathcal{M}}) = \bigoplus_{\Lambda \in \sigma(D_{\mathcal{F}})} W_{\Lambda} \quad \text{if } m \text{ even, or } m, k \text{ odd,} \quad (4.15)$$

$$\Gamma_{\mathcal{M}}(\Sigma_{\mathcal{M}}) = \bigoplus_{\Lambda \in \sigma(D_{\mathcal{F}}) \cap \mathbb{R}_0^+} W_{\Lambda} \quad \text{if } m \text{ odd, } k \text{ even.} \quad (4.16)$$

and in the same way the Dirac operator decomposes:

Proposition 4.14. For $\Lambda \in \sigma(D_{\mathcal{F}})$ we define the Hilbert space $\mathcal{H}_{\Lambda} = \mathcal{H}_{1,\Lambda} \oplus \mathcal{H}_{2,\Lambda}$ by

$$\mathcal{H}_{1,\Lambda} \oplus \mathcal{H}_{2,\Lambda} := \begin{cases} L^2(\Sigma_{\mathcal{B}}^+ \otimes \mathbb{C}^{r(\Lambda)}) \oplus L^2(\Sigma_{\mathcal{B}}^- \otimes \mathbb{C}^{r(\Lambda)}) & \text{if } m \text{ even,} \\ L^2(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}) \oplus L^2(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}) & \text{if } m \text{ odd, } k \text{ even, } \Lambda \neq 0, \\ L^2(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r_+}) \oplus L^2(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r_-}) & \text{if } m \text{ odd, } k \text{ even, } \Lambda = 0, \\ L^2(\Sigma_{\mathcal{B}}^d \otimes \mathbb{C}^{r(\Lambda)}) \oplus L^2(\Sigma_{\mathcal{B}}^{od} \otimes \mathbb{C}^{r(\Lambda)}) & \text{if } m \text{ and } k \text{ odd,} \end{cases}$$

where $r_{\pm} := \dim \ker D_{\mathcal{F}}^{\pm}$. For simplicity we write L^2 and $W^{1,2}$ for L^2 -spaces and first order Sobolev spaces when the underlying space determined by $\mathcal{H}_{1,\Lambda}$ or $\mathcal{H}_{2,\Lambda}$ is clear. We introduce the bounded operators

$$\begin{aligned} A_{1,\Lambda}: \mathcal{H}_{1,\Lambda} &\rightarrow \mathcal{H}_{1,\Lambda}, & A_{1,\Lambda} \Psi_1 &= \frac{\Lambda}{f} \Psi_1, \\ A_{2,\Lambda}: \mathcal{H}_{2,\Lambda} &\rightarrow \mathcal{H}_{2,\Lambda}, & A_{2,\Lambda} \Psi_2 &= \frac{\Lambda}{f} \Psi_2, \end{aligned}$$

and the closed operator B_Λ from $\mathcal{H}_{2,\Lambda}$ into $\mathcal{H}_{1,\Lambda}$ by

$$\mathcal{D}(B_\Lambda) = \{\Psi_2 \in \mathcal{H}_{2,\Lambda} : \Psi_2 \in W^{1,2}\},$$

$$B_\Lambda = \begin{cases} D_{\mathcal{B}}^+ \otimes I_{\mathbb{C}^{r(\Lambda)}} & \text{for } m \text{ even,} \\ D_{\mathcal{B}} \otimes I_{\mathbb{C}^{r(\Lambda)}} & \text{for } m \text{ odd, } k \text{ even, } \Lambda \neq 0, \\ iD_{\mathcal{B}} \otimes I_{\mathbb{C}^{r(\Lambda)}} & \text{for } m \text{ and } k \text{ odd.} \end{cases}$$

Then the Dirac operator $D_{\mathcal{M}}$ on the manifold \mathcal{M} can be written as

$$D_{\mathcal{M}} = \bigoplus_{\Lambda \in \sigma(D_{\mathcal{F}})} A_\Lambda$$

with self-adjoint operators A_Λ in \mathcal{H}_Λ given by the block operator representation

$$A_\Lambda = \begin{cases} \begin{pmatrix} D_{\mathcal{B}} \otimes I_{\mathbb{C}^{r_+}} & 0 \\ 0 & -D_{\mathcal{B}} \otimes I_{\mathbb{C}^{r_-}} \end{pmatrix} & \text{for } m \text{ odd, } k \text{ even and } \Lambda = 0, \\ \begin{pmatrix} A_{1,\Lambda} & B_\Lambda \\ B_\Lambda^* & -A_{2,\Lambda} \end{pmatrix} & \text{in all other cases} \end{cases}$$

with domain $\mathcal{D}(A_\Lambda) = W^{1,2} \oplus W^{1,2}$.

The proof of this proposition is completely analogous to the proof of [9, Theorem 6.1] and is therefore omitted.

Note that from the proposition it follows that eigenspaces of the Dirac operator on \mathcal{M} respect the decomposition in spinors according to (4.15) and (4.16).

Definition 4.15. An eigenvalue λ of $D_{\mathcal{M}}$ is called an *eigenvalue of weight* Λ if there is an eigenspinor Ψ associated with λ which belongs to W_Λ .

The eigenvalues of weight 0 do not depend on f and can be calculated immediately. Indeed, for $\Lambda = 0$ we have, according to Proposition 4.14,

$$A_0 = \begin{cases} \begin{pmatrix} 0 & D_{\mathcal{B}}^+ \otimes I_{\mathbb{C}^{r(0)}} \\ D_{\mathcal{B}}^- \otimes I_{\mathbb{C}^{r(0)}} & 0 \end{pmatrix} & \text{for } m \text{ even,} \\ \begin{pmatrix} D_{\mathcal{B}} \otimes I_{\mathbb{C}^{r_+}} & 0 \\ 0 & -D_{\mathcal{B}} \otimes I_{\mathbb{C}^{r_-}} \end{pmatrix} & \text{for } m \text{ odd, } k \text{ even,} \\ \begin{pmatrix} 0 & iD_{\mathcal{B}} \otimes I_{\mathbb{C}^{r(0)}} \\ -iD_{\mathcal{B}} \otimes I_{\mathbb{C}^{r(0)}} & 0 \end{pmatrix} & \text{for } m \text{ and } k \text{ odd.} \end{cases}$$

Hence, disregarding multiplicities, for m even we have $\sigma(A_0) = \sigma(D_{\mathcal{B}})$ and for m odd we have $\sigma(A_0) = \sigma(D_{\mathcal{B}}) \cup \sigma(-D_{\mathcal{B}})$.

To get estimates for the eigenvalues of non-vanishing weight, we need the following proposition, which follows by careful enumeration of the eigenvalues.

Proposition 4.16. Define the dimension of the kernel of $D_{\mathcal{B}}$ and the Fredholm index of $D_{\mathcal{B}}^+$ by

$$\gamma := \dim \ker D_{\mathcal{B}}, \quad \delta := \text{ind } D_{\mathcal{B}}^+ = \dim \ker D_{\mathcal{B}}^+ - \dim \ker D_{\mathcal{B}}^-,$$

and set

$$\alpha := \frac{\gamma - \delta}{2} = \dim \ker D_{\mathcal{B}}^-, \quad \beta := \frac{\gamma + \delta}{2} = \dim \ker D_{\mathcal{B}}^+.$$

Let ζ_n , $n = 1, 2, \dots$, be the eigenvalues of $D_{\mathcal{B}}$ counted with multiplicities and enumerated such that the sequence $(|\zeta_n|)$ is non-decreasing and, if m is even, $\zeta_{\gamma+j} < 0$ for j odd.

Then the eigenvalues $\nu_{1,\Lambda,n}$ and $\nu_{2,\Lambda,n}$, $n = 1, 2, \dots$, of $(B_{\Lambda} B_{\Lambda}^*)^{1/2}$ and of $(B_{\Lambda}^* B_{\Lambda})^{1/2}$, respectively, enumerated non-decreasingly, are given by

i) for m even:

$$\begin{aligned} \nu_{1,\Lambda,n} &= 0, & n &= 1, \dots, \alpha r(\Lambda), \\ \nu_{1,\Lambda,n} &= \zeta_{\gamma+2j}, & n &= \alpha r(\Lambda) + (j-1)r(\Lambda) + 1, \dots, \alpha r(\Lambda) + jr(\Lambda), \\ \nu_{2,\Lambda,n} &= 0, & n &= 1, \dots, \beta r(\Lambda), \\ \nu_{2,\Lambda,n} &= \nu_{1,\Lambda,n+\delta r(\Lambda)} & n &= \beta r(\Lambda) + 1, \beta r(\Lambda) + 2, \dots, \end{aligned}$$

ii) for m odd:

$$\nu_{1,\Lambda,n} = \nu_{2,\Lambda,n} = |\zeta_j| \quad n = (j-1)r(\Lambda) + 1, \dots, jr(\Lambda).$$

Note that for m even, disregarding multiplicities, we have the identities

$$\{\nu_{1,\Lambda,n}\} \cup \{-\nu_{2,\Lambda,n}\} = \sigma(D_{\mathcal{B}}),$$

and in any case we have

$$\{\nu_{1,\Lambda,n}\} \cup \{-\nu_{2,\Lambda,n}\} = \sigma(D_{\mathcal{B}}) \cup \sigma(-D_{\mathcal{B}}).$$

Applying Corollary 4.8 to the operators \mathcal{A}_{Λ} we obtain the following theorem:

Theorem 4.17. *Let $\mathcal{M} = \mathcal{B}^m \times_f \mathcal{F}^k$ be a warped product of closed spin manifolds and denote by f_{\max} and f_{\min} the maximum and minimum of $f: \mathcal{B}^m \rightarrow \mathbb{R}^+$, respectively. Let $\nu_{1,\Lambda,n}$ and $\nu_{2,\Lambda,n}$, $n = 1, 2, \dots$, be the eigenvalues of $(B_{\Lambda} B_{\Lambda}^*)^{1/2}$ and of $(B_{\Lambda}^* B_{\Lambda})^{1/2}$, respectively, given according to Proposition 4.16. Then the eigenvalues $\lambda_{\Lambda,n}$, $n = \pm 1, \pm 2, \dots$, of $D_{\mathcal{M}}$ of weight $\Lambda \neq 0$, $\Lambda \in \sigma(D_{\mathcal{F}})$, i.e., the eigenvalues of \mathcal{A}_{Λ} , can be enumerated such that $\dots \leq \lambda_{\Lambda,-2} \leq \lambda_{\Lambda,-1} < 0 < \lambda_{\Lambda,1} \leq \lambda_{\Lambda,2} \leq \dots$ and, for $n = 1, 2, \dots$,*

$$\lambda_{\Lambda,n} \geq -\frac{|\Lambda|}{2} \left(\frac{1}{f_{\min}} - \frac{1}{f_{\max}} \right) + \sqrt{\frac{\Lambda^2}{4} \left(\frac{1}{f_{\min}} + \frac{1}{f_{\max}} \right)^2 + \nu_{1,\Lambda,n}^2},$$

$$\lambda_{\Lambda,n} \leq \frac{|\Lambda|}{2} \left(\frac{1}{f_{\min}} - \frac{1}{f_{\max}} \right) + \sqrt{\frac{\Lambda^2}{4} \left(\frac{1}{f_{\min}} + \frac{1}{f_{\max}} \right)^2 + \nu_{1,\Lambda,n}^2},$$

and

$$\lambda_{\Lambda,-n} \geq -\frac{|\Lambda|}{2} \left(\frac{1}{f_{\min}} - \frac{1}{f_{\max}} \right) - \sqrt{\frac{\Lambda^2}{4} \left(\frac{1}{f_{\min}} + \frac{1}{f_{\max}} \right)^2 + \nu_{2,\Lambda,n}^2},$$

$$\lambda_{\Lambda,-n} \leq \frac{|\Lambda|}{2} \left(\frac{1}{f_{\min}} - \frac{1}{f_{\max}} \right) - \sqrt{\frac{\Lambda^2}{4} \left(\frac{1}{f_{\min}} + \frac{1}{f_{\max}} \right)^2 + \nu_{2,\Lambda,n}^2}.$$

Note that if Λ were equal to 0 in the above estimates, upper and lower bounds are the same and coincide with the eigenvalues of weight 0, apart from multiplicities.

Corollary 4.18. *For all eigenvalues of weight $\Lambda \neq 0$ we have*

$$|\lambda_{\Lambda,n}| \geq \frac{|\Lambda|}{f_{\max}}, \quad n = \pm 1, \pm 2, \dots \quad (4.17)$$

If $0 \in \sigma(D_{\mathcal{B}})$, then for the eigenvalue $\lambda_{\Lambda,\min}$ of weight $\Lambda \neq 0$ with smallest modulus, we have

$$|\lambda_{\Lambda,\min}| \leq \frac{|\Lambda|}{f_{\min}}. \quad (4.18)$$

Proof. The estimate (4.17) follows from the estimates in Theorem 4.17 by omitting the terms $\nu_{1,\Lambda,n}^2$ and $\nu_{2,\Lambda,n}^2$. If $0 \in \sigma(D_{\mathcal{B}})$, then either $\nu_{1,\Lambda,1} = 0$ or $\nu_{2,\Lambda,1} = 0$, which implies (4.18). \square

Remark 4.19. The Dirac operator on the warped product is not only symmetric in the case where the dimension of \mathcal{M} is even but additionally in all cases where the spectrum of the fibre \mathcal{F} is symmetric. This arises from different phenomena in the cases m odd and m even:

If m is odd, then the spectrum of fixed weight Λ is symmetric. More exactly, if λ is an eigenvalue of \mathcal{A}_{Λ} with eigenvector $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, then $-\lambda$ is an eigenvalue of \mathcal{A}_{Λ} with eigenvector $\begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix}$ if k is even and with eigenvector $\begin{pmatrix} J^{-1}\psi_2 \\ J\psi_1 \end{pmatrix}$ if k is odd. Here J is the canonical isomorphism from $L^2(\Sigma_{\mathcal{B}}^d)$ to $L^2(\Sigma_{\mathcal{B}}^{od})$, which has the property $J D_{\mathcal{B}} = D_{\mathcal{B}} J^{-1}$.

If m is even, then the spectrum of fixed weight Λ is not symmetric, but if the spectrum of the fibre \mathcal{F} is symmetric and $\lambda \in \sigma_{\mathbb{P}}(D_{\mathcal{M}})$ is an eigenvalue of \mathcal{A}_{Λ} with eigenvector $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, then $-\lambda$ is an eigenvalue of $\mathcal{A}_{-\Lambda}$ with eigenvector $\begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}$ since $-A_{i,\Lambda} = A_{i,-\Lambda}$, $i = 1, 2$, and thus also $-\lambda \in \sigma_{\mathbb{P}}(D_{\mathcal{M}})$.

From Remark 4.10 the following corollary follows immediately.

Corollary 4.20. *If \mathcal{B} is a Riemannian spin manifold with a parallel spinor Ψ (i.e., $\nabla\Psi = 0$), then for the first positive eigenvalue $\lambda_{\Lambda,1}$ of the Dirac operator on $\mathcal{B} \times_f \mathcal{F}$ of weight Λ we have*

$$\frac{|\Lambda|}{f_{\max}} \leq \lambda_{\Lambda,1} \leq \frac{1}{\text{vol } \mathcal{B}} \int_{\mathcal{B}} \frac{|\Lambda|}{f} d\mathcal{B}.$$

Riemannian spin manifolds with parallel spinors have been classified in [19] and [18]. In dimension 3, Riemannian spin manifolds admitting a parallel spinor are flat, in dimension 4, for instance, a compact spin manifold with a nontrivial parallel spinor is flat or a $K3$ surface with Yau metric [6].

Special examples are the warped products $\mathcal{M} = S^1 \times_f \mathcal{F}$ with a closed manifold \mathcal{F} . In this case we consider $\Sigma_{S^1} = S^1 \times \mathbb{C}$ and, e.g., if $\dim \mathcal{F}$ is odd, the Dirac operator $D_{\mathcal{M}}$ is given by $D_{\mathcal{M}} = \bigoplus \mathcal{A}_{\Lambda}$, with

$$\mathcal{A}_{\Lambda} = \begin{pmatrix} \Lambda/f & -d/dx \\ d/dx & -\Lambda/f \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{\Lambda}) = W^{1,2}(S^1, \mathbb{C}^{r(\Lambda)}) \oplus W^{1,2}(S^1, \mathbb{C}^{r(\Lambda)}),$$

which has been considered in Example 4.11. Dirac operators on such manifolds have been studied intensively in [8]. In this simpler case, where the basis manifold is one-dimensional, ODE methods are available and so it was possible to derive better asymptotic estimates using Floquet theory.

References

- [1] F. V. Atkinson, H. Langer, R. Mennicken, A. A. Shkalikov: The essential spectrum of some matrix operators, *Math. Nachr.* **167** (1994), 5–20.
- [2] D. Eschwé, M. Langer: Variational principles for eigenvalues of self-adjoint operator functions, *Integral Equations Operator Theory*, to appear.
- [3] J. Dolbeault, M. J. Esteban, E. Séré: On the eigenvalues of operators with gaps. Application to Dirac operators, *J. Funct. Anal.* **174** (2000), 208–226.
- [4] Th. Friedrich: *Dirac-Operatoren in der Riemannschen Geometrie*, Vieweg 1997, Braunschweig, Wiesbaden.
- [5] M. Griesemer, H. Siedentop: A minimax principle for the eigenvalues in spectral gaps, *J. London Math. Soc. (2)* **60** (1999), 490–500.
- [6] N. Hitchin: Compact Four-Dimensional Einstein Manifolds, *J. Diff. Geom.* **9** (1974), 435–441.
- [7] T. Kato: *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin – Heidelberg – New York, 1976.
- [8] M. Kraus: Asymptotic estimates for the eigenvalues of the Dirac and the Laplace operator on warped products over S^1 , submitted.
- [9] M. Kraus, C. Tretter: A new method for eigenvalue estimates for Dirac operators on certain manifolds with S^k -symmetry, *Differential Geom. Appl.*, to appear.
- [10] H. Langer, M. Langer, C. Tretter: Variational principles for eigenvalues of block operator matrices, *Indiana Univ. Math. J.* **51** (2002), 1427–1459.
- [11] H. Langer, A. Markus, V. Matsaev, C. Tretter: A new concept for block operator matrices: the quadratic numerical range, *Linear Algebra Appl.* **330** (2001), 89–112.
- [12] H. Langer, C. Tretter: Spectral decomposition of some nonselfadjoint block operator matrices, *J. Operator Theory* **39** (1998), 339–359.
- [13] H. Langer, C. Tretter: Diagonalization of certain block operator matrices and applications to Dirac operators, *Oper. Theory Adv. Appl.* **122** (2001), 331–358.
- [14] H. B. Lawson, M.-L. Michelsohn: *Spin Geometry*, Princeton University Press, Princeton (1989).
- [15] A. E. Lifschitz: *Magnetohydrodynamics and Spectral Theory*, Kluwer Academic Publishers Group, Dordrecht, 1989.
- [16] G. D. Raikov: The spectrum of a linear magnetohydrodynamic model with cylindrical symmetry, *Arch. Rational Mech. Anal.* **116** (1991), 161–198.
- [17] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York – London, 1978.

- [18] M. Y. Wang: Parallel spinors and parallel forms, *Ann. Glob. Anal. Geom.* **7** (1989), 59–68.
- [19] M. Y. Wang: On non-simply connected manifolds with non-trivial parallel spinors, *Ann. Glob. Anal. Geom.* **13** (1995), 31–42.
- [20] A. Weinstein, W.F. Stenger: *Methods of Intermediate Problems for Eigenvalues. Theory and Ramifications*, Mathematics in Science and Engineering 89, Academic Press, New York – London, 1972.

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